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1992 J. Phys. A: Math. Gen. 25 2181

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## Critical exponents for $c$ -animals with nearest-neighbour interaction

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Received 30 July 1991, in final form 9 December 1991

**Abstract.** We study two lattice models: (a)  $c$ -animals with an interaction energy  $\alpha$  between nearest-neighbour pairs of vertices; (b)  $c$ -animals as in (a) but with an additional interaction energy  $\omega$  between the vertices of the animal and an adsorption surface. By assuming that the partition functions satisfy  $A_n(c, \alpha) \sim n^{-\theta_c(\alpha)} \lambda_c(\alpha)^n$  and  $A_n(c, \alpha, \omega) \sim n^{-\theta_c(\alpha, \omega)} \lambda_c(\alpha, \omega)^n$  as  $n \rightarrow \infty$  with  $c$  fixed, we show that  $\theta_c(\alpha) = \theta_0(\alpha) - c$ ,  $-\infty < \alpha < \infty$  and  $\theta_c(\alpha, \omega) = \theta_0(\alpha, \omega) - c$ ,  $-\infty < \alpha, \omega < \infty$ , where  $\theta_0(\alpha)$  and  $\theta_0(\alpha, \omega)$  are the corresponding exponents for trees.

### 1. Introduction

A (weakly embedded)  $c$ -animal is a connected subgraph on a regular  $d$ -dimensional lattice with a cyclomatic index  $c$ . The cyclomatic index is the number of independent cycles, or the maximum number of edges which can be removed without disconnecting the animal. When  $c = 0$ , the animal is called a tree. The number of  $c$ -animals with  $n$  vertices  $a_n(c)$ , is expected to have the asymptotic behaviour

$$a_n(c) \sim n^{-\theta_c} \lambda_c^n \quad (1)$$

where  $\lambda_c$  is the connective constant of the lattice. It has been shown (Whittington *et al* 1983) that  $\lambda_c$  is independent of the cyclomatic index  $c$ . For the exponent  $\theta_c$  for the subdominant term, Soteros and Whittington (1988) have established

$$\theta_c = \theta_0 - c \quad (2)$$

if either of the exponents exist.

In a previous paper, we have considered the interaction of a  $c$ -animal with an adsorption surface through an interaction energy  $\omega$ . By assuming the existence of  $\theta_0(\omega)$ , we have shown that  $\theta_c(\omega)$  also exists and satisfies

$$\theta_c(\omega) = \theta_0(\omega) - c \quad (3)$$

for  $-\infty < \omega < +\infty$  (Zhao and Lookman 1991).

These results have been obtained by considering  $c$ -animals in a good solvent without an interaction between nearest-neighbour pairs of vertices. Recently, branched polymers in which the monomers are allowed to interact with each other and which undergo a collapse transition, have received some attention (Whittington *et al* 1991, Madras *et al* 1991 and references there). Such a problem is modelled by considering lattice animals in which there is an attractive interaction energy  $\alpha$  between a pair of vertices

which are a unit distance apart but are not incident on a common edge (hereafter, we refer to such a pair as a nearest-neighbour contact or simply a contact).

In this paper, we consider both  $c$ -animals with a nearest-neighbour interaction and  $c$ -animals with both a nearest-neighbour interaction and an interaction with an adsorption surface. The partition functions for the two models are defined as

$$A_n(c, \alpha) = \sum_i a_{n,i}(c) e^{i\alpha} \tag{4}$$

(Whittington *et al* 1991) and

$$A_n(c, \alpha, \omega) = \sum_{i,j} a_{n,i,j}(c) e^{i\alpha + j\omega} \tag{5}$$

(Veal *et al* 1990) where  $a_{n,i}(c)$  is the number of  $n$ -vertex  $c$ -animals with  $i$  contacts and  $a_{n,i,j}$  is the number of  $n$ -vertex  $c$ -animals with  $i$  contacts and  $j$  vertices in the surface. From the arguments of Whittington *et al* (1991), Madras *et al* (1991) and Lookman *et al* 1990, one can show that the limits

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln A_n(c, \alpha) = \ln \lambda_0(\alpha) \tag{6}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln A_n(c, \alpha, \omega) = \ln \lambda_0(\alpha, \omega) \tag{7}$$

exist for all  $\alpha$  and  $\omega$ , and are independent of the cyclomatic index  $c$  with  $\ln \lambda_0(\alpha, \omega)$  satisfying

$$\max\{\ln \lambda_0(\alpha), \ln \lambda_0^*(\alpha) + \omega\} \leq \ln \lambda_0(\alpha, \omega) \leq \max\{\ln \lambda_0(\alpha), \ln \lambda_0(\alpha) + \omega\} \tag{8}$$

where  $\ln \lambda_0^*(\alpha)$  is the limit (6) of the corresponding  $(d - 1)$ -dimensional lattice. By analogy, one can expect both  $A_n(c, \alpha)$  and  $A_n(c, \alpha, \omega)$  to have asymptotic behaviour similar to that in (1) with

$$\theta_c(\alpha) = \theta_0(\alpha) - c \quad \text{for } -\infty < \alpha < \infty \tag{9}$$

and

$$\theta_c(\alpha, \omega) = \theta_0(\alpha, \omega) - c \quad \text{for } -\infty < \alpha \quad \omega < \infty. \tag{10}$$

We prove these two equalities by assuming that  $\theta_0(\alpha)$  and  $\theta_0(\alpha, \omega)$  exist. We extend the local transformation given by Soteros and Whittington (1988) (referred to as sw) for branch points (vertices with degree greater than 2) to vertices with at least one contact and vertices in the surface. This results in the inequalities

$$A_{n+2c}(c, \alpha) \geq f(\alpha) n^c T_n(\alpha) \tag{11}$$

and

$$A_{n+3c}(c, \alpha, \omega) \geq f'(\alpha, \omega) n^c T_n(\alpha, \omega) \tag{12}$$

where  $T_n(\alpha)$  and  $T_n(\alpha, \omega)$  are the corresponding partition functions for trees. We note that deleting  $c$  edges from a  $c$ -animal without disconnecting it yields  $c$  extra nearest-neighbour contacts but does not change the number of vertices on the adsorption surface. Therefore, from an argument given by Whittington *et al* (1983), one can derive

$$A_n(c, \alpha) \leq (2dn)^c e^{c|\alpha|} T_n(\alpha) \tag{13}$$

and

$$A_n(c, \alpha, \omega) \leq (2dn)^c e^{c|\alpha|} T_n(\alpha, \omega). \tag{14}$$

Together, these inequalities give (9) and (10).

For convenience, we will concentrate on  $c$ -animals embedded in the square lattice where the 'surface' is the line  $x = 0$ . However, the generalization of the corresponding results to higher-dimensional lattices is straightforward.

**2. *c*-animals with nearest-neighbour interaction**

This section is concerned with weak embeddings in the square lattice with only nearest-neighbour interaction.

*2.1.*

In the square lattice, an  $n$ -vertex tree  $T$  consists of  $n$  connected vertices with coordinates  $(x_i, y_i)$ ,  $i = 1, \dots, n$ . The top vertex  $v_t$  (bottom vertex  $v_b$ ) of the tree is defined as the vertex with maximum (minimum)  $x$  coordinate and in turn the vertex in this subset having maximum (minimum)  $y$  coordinate. A contact consists of a pair of vertices  $v'$ ,  $v''$  which are nearest neighbours but not connected to each other by an edge. Hereafter, for such a pair, we say that  $v'$  has a contact with  $v''$  if  $x' + y' < x'' + y''$  and vice versa. Thus, a vertex  $v_0 = (x, y)$  can only have contacts with the vertices  $v_1 = (x + 1, y)$  and  $v_2 = (x, y + 1)$ . We classify such a vertex  $v_0$  of a tree by:

- (a)  $v_0$  is in the set  $\mathcal{V}_1$  if it has a contact with  $v_1$  and is connected to  $v_2$ .
- (b)  $v_0$  is in the set  $\mathcal{V}_2$  if it has a contact with  $v_2$  and is connected to  $v_1$ .
- (c)  $v_0$  is in the set  $\mathcal{V}_3$  if it has a contact with  $v_1$ , but  $v_2$  is not in the tree.
- (d)  $v_0$  is in the set  $\mathcal{V}_4$  if it has a contact with  $v_2$ , but  $v_1$  is not in the tree.
- (e)  $v_0$  is in the set  $\mathcal{V}_5$  if it has contacts with both  $v_1$  and  $v_2$ .

*Theorem 1.* Every tree  $T$  with a vertex  $v_0 \in \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4$  or  $\mathcal{V}_5$  can be converted into an  $(n + 2)$ -vertex 1-animal containing a 4-cycle in which  $v_0$  is the bottom vertex of the 4-cycle. The resulting animal can have at most  $14 \times 3$  such trees as its precursors.

*Proof.* We consider a tree  $T$  which has a vertex  $v_0 = (x, y) \in \mathcal{V}_1$ . Then,  $v_0$  has a contact with  $v_1$  and is connected to  $v_2$ . The tree must contain at least one of the vertices  $v_{11} = (x + 1, y + 1)$ ,  $v_{12} = (x + 2, y)$  and  $v_{13} = (x + 1, y - 1)$  and one of the edges  $[v_1, v_{11}]$ ,  $[v_1, v_{12}]$  and  $[v_1, v_{13}]$ .  $v_0$  is connected to  $v_1$  through one and only one of these three edges. Delete such an edge on this connected path and add the edge  $[v_0, v_1]$  and the vertex  $v'_i = (x_i + 1, y_i)$  and the edge  $[v_i, v'_i]$  if  $[v_1, v_{12}]$  is deleted. Otherwise, add the vertex  $v''_i = (x_i, y_i + 1)$  and the edge  $[v_i, v''_i]$  if either  $[v_1, v_{11}]$  or  $[v_1, v_{13}]$  is deleted. The resulting connected subgraph has  $n + 1$  vertices and  $n$  edges. It is a tree  $T'$  with  $v_0$  connected to both  $v_1$  and  $v_2$ . Since the constructions for trees with  $v_0$  connected to  $v_1$  through either  $[v_1, v_{11}]$  or  $[v_1, v_{13}]$  are the same, there can be at most two trees containing a vertex in  $\mathcal{V}_1$  that is mapped to the same  $T'$ .

Similarly, with a little modification of the above procedure, a tree with a vertex  $v_0$  in one of the other four sets can be converted it into a  $(n + 1)$ -vertex tree  $T'$  such that  $v_0$  is connected to both  $v_1$  and  $v_2$ . One can show that the resulting tree,  $T'$  can have, as its precursors, at most 2 trees with  $v_0 \in \mathcal{V}_2$ , 3 trees with  $v_0 \in \mathcal{V}_3$ , 3 trees with  $v_0 \in \mathcal{V}_4$  and 4 trees with  $v_0 \in \mathcal{V}_5$ . Therefore, together there can be at most 14 such trees that are mapped to the same  $T'$ .

Then the transformation given in sw in theorem 1 is applied to  $T'$  at  $v_0$ . This results in converting  $T'$  into an  $n + 2$ -vertex 1-animal which has a 4-cycle at  $v_0$ , consisting of the vertices  $v_0, v_1, v_2$  and  $v_3 = (x + 1, y + 1)$  and the edges  $[v_0, v_1]$   $[v_1, v_3]$ ,  $[v_3, v_2]$  and  $[v_2, v_0]$ . From the same theorem, the resulting animal can have at most such trees,  $T'$ , as precursors. Hence, the resulting animal can have at most  $14 \times 3$  trees,  $T$ , as its precursors.

Consider a tree  $T$  containing more than  $\varepsilon n$  ( $\varepsilon > 0$ ) contacts. Since a vertex can have at most two contacts,  $T$  must have at least  $\varepsilon n/2$  vertices in  $\mathcal{V}_1 \cup \dots \cup \mathcal{V}_5$ . Therefore, there can be at least

$$\binom{\varepsilon n/2}{1} \tag{15}$$

ways to choose  $v_0$ . We note that, in the above procedure, there is a neighbourhood,  $\mathcal{N}(v_0)$ , of  $v_0$  defined by

$$\mathcal{N}(v_0): \quad |x - x_0| \leq 1 \quad |y - y_0| \leq 1 \tag{16}$$

such that if  $T$  has another vertex  $v'_0$  which is in  $\mathcal{V}_1 \cup \dots \cup \mathcal{V}_5$  but not in  $\mathcal{N}(v_0)$ , repeating the same transformation at  $v'_0$  will leave the 4-cycle at  $v_0$  unchanged.  $T$  can have at most 9 vertices contained in  $\mathcal{N}(v_0)$  and these vertices can have at most 18 contacts. Hence, if  $\varepsilon n > (c - 1) \times 18$ , there are at least

$$\binom{\varepsilon n/2}{1} \binom{(\varepsilon n - 18)/2}{1} \dots \binom{(\varepsilon n - (c - 1)18)/2}{1} / c! \tag{17}$$

ways to select the  $c$  vertices at which we can carry out the transformation successfully to convert  $T$  into a  $c$ -animal.

2.2.

Let  $\mathcal{T}_n$  be the set of  $n$ -vertex trees. We define  $\mathcal{T}_n^{(1)}(\varepsilon, >)$  to be the set of  $n$ -vertex trees containing more than  $\varepsilon n$  branch points,  $\mathcal{T}_n^{(2)}(\varepsilon, >)$  to be the set of  $n$ -vertex trees containing more than  $\varepsilon n$  contacts and

$$\mathcal{T}_n(\varepsilon, >) = \mathcal{T}_n^{(1)}(\varepsilon, >) \cup \mathcal{T}_n^{(2)}(\varepsilon, >). \tag{18}$$

For a tree  $T \in \mathcal{T}_n^{(1)}(\varepsilon, >)$ , we convert it into a  $c$ -animal with  $n + c$  vertices by following the argument in sw. We note that the resulting animal can have at most  $2c$  extra contacts. Denoting by  $t_{n,i}^{(1)}$  the number of trees in  $\mathcal{T}_n^{(1)}(\varepsilon, >)$  containing  $i$  contacts, we have

$$a_{n+c,i}(c) + a_{n,i+1}(c) + \dots + a_{n,i+2c}(c) \geq \binom{\varepsilon' n}{c} t_{n,i}^{(1)} / 2 \times 3^c \tag{19}$$

with  $\varepsilon' = \varepsilon/5$ . Multiplying both sides by  $e^{i\alpha}$  and summing over  $i$  gives

$$c(1 + e^{|\alpha|} + \dots + e^{2c|\alpha|}) A_{n+c}(c, \alpha) \geq \binom{\varepsilon' n}{c} T_n^{(1)}(\varepsilon, >, \alpha) \tag{20}$$

or simply

$$A_{n+c}(c, \alpha) \geq f_1(\alpha) (\varepsilon' n)^c (1 + o(1)) T_n^{(1)}(\varepsilon, >, \alpha). \tag{21}$$

A tree  $T \in \mathcal{T}_n^{(2)}(\varepsilon, >)$ , has more than  $\varepsilon n$  contacts. By following the above procedure, we convert it into a  $c$ -animal which can have at most  $4c$  extra contacts. Let  $t_{n,i}^{(2)}$  be the number of trees in  $\mathcal{T}_n^{(2)}(\varepsilon, >)$  containing  $i$  contacts, from the above argument, we have

$$a_{n+2c,i}(c) + \dots + a_{n+2c,4c}(c) \geq \prod_{i=1}^c \binom{\varepsilon n - (i - 1) \times 18}{1} t_{n,i}^{(2)} / C \tag{22}$$

where  $C = (14 \times 3)^c c!$  Multiplying both sides with  $e^{i\alpha}$  and summing over  $i$  gives

$$c(1 + e^{|\alpha|} + \dots + e^{4c|\alpha|})A_{n+2c}(c, \alpha) \geq \prod_{i=1}^c \binom{\epsilon n - (i-1) \times 18}{1} T_n^{(2)}(\epsilon, >, \alpha) / C \tag{23}$$

or

$$A_{n+2c}(c, \alpha) \geq f_2(\alpha)(\epsilon n)^c (1 + o(1)) T_n^{(2)}(\epsilon, >, \alpha). \tag{24}$$

From equations (21) and (24), we have

$$A_{n+2c}(\epsilon, >) \geq f(\alpha)(\epsilon' n)^c T_n(\epsilon, >, \alpha) \tag{25}$$

where we used the fact that  $A_{2n+2c}(c, \alpha) \geq A_{n+c}(c, \alpha)$  and  $T_n(\epsilon, >, \alpha)$  is the partition function for trees in the set  $\mathcal{T}_n(\epsilon, >)$ .

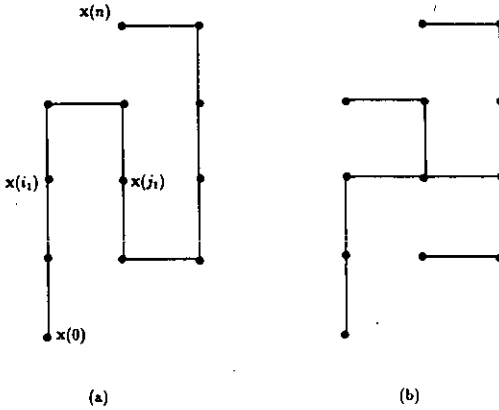
### 2.3.

We classify contacts and describe a transformation to convert a walk into a tree which satisfies certain requirements. We define a contact to be a type-I contact if two vertices forming the contact belong to the same branch and at least one of them has degree 2. Otherwise, we define the contact to be a type-II contact. We denote by  $\mathcal{T}'_n(\epsilon, \leq)$  the set of  $n$ -vertex trees containing at most  $\epsilon n$  branch points and at most  $\epsilon n$  type-II contacts and  $\mathcal{W}_n(\epsilon, \leq)$  the set of  $n$ -step self-avoiding walks with at most  $\epsilon n$  contacts.

*Lemma 1.* Every walk  $W \in \mathcal{W}_n(\epsilon, \leq)$  can be converted into a tree  $T' \in \mathcal{T}'_{n+1}(\epsilon, \leq)$  such that if  $W$  has  $k$  contacts,  $T'$  has  $k$  II-type contacts and at most  $k$  branch points.  $T'$  can have at most  $9^k$  walks as its precursors.

*Proof.* Let  $W = \{x(0), x(1), \dots, x(n)\}$  be an  $n$ -step self-avoiding walk. If  $W$  has no contact or has only one contact consisting of  $x(0)$  and  $x(n)$ , which is a type-II contact from the definition, the walk is a member of  $\mathcal{T}'_{n+1}(\epsilon, \leq)$ . We therefore consider a walk  $W$  with  $k (\geq 1)$  contacts with at least one type-I contact. Starting at  $x(0)$ , we move along the walk and stop at  $x(j_1)$  which is the first vertex in contact with one of the vertices  $x(0), \dots, x(j_1 - 1)$ , named  $x(i_1)$ . If  $j_1 = n$ ,  $W$  has only one type-I contact. We add the edge  $[x(i_1), x(j_1)]$  and delete the edge  $[x(j_1 - 1), x(j_1)]$ , which converts the walk into a tree with a branch point of degree 3 at the vertex  $x(i_1)$  and one type-II contact. If  $j_1 < n$ , we add the edge  $[x(i_1), x(j_1)]$  and delete the edge  $[x(i_1 + 1), x(i_1 + 2)]$ , which gives a tree with one branch point of degree 3 at the vertex  $x(j_1)$  and  $k$  contacts of which at least one is a type-II contact. Obviously, the resulting tree can only have type-I contacts on the branch  $\{x(j_1), x(j_1 + 1), \dots, x(n)\}$ . Next, we examine the walk  $\{x(j_1), x(j_1 + 1), \dots, x(n)\}$  and repeat the same procedure along it. Since  $W$  can have at most  $k$  type-I contacts, by repeating the procedure at most  $k$  times, we convert  $W$  into an  $(n+1)$ -vertex tree with  $k$  branch points of degree 3 and exactly  $k$  type-II contacts (figure 1).

Let  $T'$  be a tree constructed following the above procedure with only one branch point  $v$  of degree 3. There then exists at least one vertex  $v'$  such that it is connected to  $v$  by the edge  $[v, v']$  and in contact with the end vertex  $v''$  of one branch. We add the edge  $[v', v'']$  and delete the edge  $[v, v']$ , which gives an  $n$ -step self-avoiding walk which can be converted into  $T'$  by the above procedure. Since  $v$  has degree 3, it is



**Figure 1.** (a) An 11-step self-avoiding walk with five type-I nearest-neighbour contacts; (b) a 12-vertex tree constructed from the walk (a) with two (<5) branch points and exactly five type-II nearest-neighbour contacts.

connected to 3 nearest-neighbour vertices. Each of these 3 vertices can be in contact with the end vertex of at most 3 branches. Therefore, there are at most  $(3 \times 3)$  ways of combining together to choose  $v'$  and  $v''$  and each of them will give a different  $n$ -step self-avoiding walk. Hence, there can exist at most 9 walks as precursors for  $T'$ . If  $T'$  has  $k$  branch points, from the above argument, there can exist at most  $9^k$  walks as the precursors for  $T'$ .

2.4.

We define  $\mathcal{T}_n(\varepsilon, \leq) = \mathcal{T}_n \setminus \mathcal{T}_n(\varepsilon, >)$ , which is the set of  $n$ -vertex trees containing at most  $\varepsilon n$  branch points and at most  $\varepsilon n$  contacts. We denote by  $t_{n,i}(\varepsilon, \leq)$  the number of trees in  $\mathcal{T}_n(\varepsilon, \leq)$  with exactly  $i$  contacts. By an argument analogous to that of Lipson and Whittington (1983), one can show that the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln T_n(\varepsilon, \leq, \alpha) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{\varepsilon n} t_{n,i}(\varepsilon, \leq) e^{i\alpha} = \ln \lambda_0(\varepsilon, \alpha) \tag{26}$$

exists for all  $\alpha$ .

When  $\varepsilon = 0$ , a member in  $\mathcal{T}_n(0, \leq)$  reduces to an  $(n - 1)$ -step self-avoiding walk without any nearest-neighbour contact. By a standard procedure (Whittington 1988), one can show the existence of the limit

$$\ln \lambda_0(0, \alpha) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln T_n(0, \leq, \alpha) = \kappa'. \tag{27}$$

From lemma 1 and arguments in sw, we obtain theorem 2.

**Theorem 2.** For given  $\alpha$ , the function of  $\ln \lambda_0(\varepsilon, \alpha)$  is continuous at  $\varepsilon = 0$ . That is

$$\lim_{\varepsilon \rightarrow 0} \ln \lambda_0(\varepsilon, \alpha) = \ln \lambda_0(0, \alpha) = \kappa'. \tag{28}$$

By using a result given by Whittington *et al* (1991) and an argument analogous to that of Gaunt *et al* (1982), one can show that, for any  $\alpha$ ,

$$\ln \lambda_0(\alpha) > \kappa'. \tag{29}$$

From (28) and (29), we have.

**Lemma 2.** For given  $\alpha$ , there exists  $\varepsilon_0(\alpha) > 0$  such that for any  $\varepsilon < \varepsilon_0(\alpha)$ ,

$$\lim_{n \rightarrow \infty} \frac{T_n(\varepsilon, >, \alpha)}{T_n(\alpha)} = 1. \tag{30}$$

With the lemma 2 and equations (13) and (25), we obtain

**Theorem 3.** If  $\lim_{n \rightarrow \infty} [\ln T_n(\alpha) - n \ln \lambda_0(\alpha)] / \ln n = -\theta_0(\alpha)$  exists, then

$$\lim_{n \rightarrow \infty} \frac{\ln A_n(c, \alpha) - n \ln \lambda_0(\alpha)}{\ln n} = -\theta_c(\alpha) \tag{31}$$

exists for all  $c$  and

$$\theta_c(\alpha) = \theta_0(\alpha) - c. \tag{32}$$

### 3. The presence of an adsorption surface

In this section, we consider *c*-animals rooted at an adsorption surface with both the nearest-neighbour interaction and the interaction with the surface. The surface can be either penetrable or impenetrable. In the following, we do not distinguish the surfaces. However, the appropriate results apply, depending on the surface.

Let  $\mathcal{T}_n$  be the set of  $n$ -vertex trees rooted at the surface. For  $\varepsilon > 0$ , the subsets  $\mathcal{T}_n^{(1)}(\varepsilon, >)$  and  $\mathcal{T}_n^{(2)}(\varepsilon, >)$  are the same as defined in section 2. By following the same procedure there and an argument of Zhao and Lookman (1991), we can convert the trees in these subsets into *c*-animals and establish

$$A_{n+c}(c, \alpha, \omega) \geq f_1(\alpha, \omega)(\varepsilon'n)^c T_n^{(1)}(\varepsilon, >, \alpha, \omega) \tag{33}$$

and

$$A_{n+2c}(c, \alpha, \omega) \geq f_2(\alpha, \omega)(\varepsilon n)^c T_n^{(2)}(\varepsilon, >, \alpha, \omega) \tag{34}$$

with  $\varepsilon' = \varepsilon/5$ .

We define the third subset  $\mathcal{T}_n^{(3)}(\varepsilon, >)$  of  $\mathcal{T}_n$ , which is the set of all  $n$ -vertex trees with more than  $\varepsilon(n-1)$  vertices in the surface. For a tree  $T \in \mathcal{T}_n^{(3)}(\varepsilon, >)$ , we choose one vertex of the tree,  $v_0 = (0, y)$  which is in the surface, and consider all possible configurations of the vertex with the vertices  $v_1 = (1, y)$  and  $v_2 = (0, y+1)$  (there are a total of 9 such configurations). By following a procedure similar to that given in theorem 1, we convert the tree into an  $(n+3)$ -vertex 1-animal with a 4-cycle in which  $v_0$  is the bottom vertex of the cycle. It can be shown that the resulting animal can have a finite number of trees as its precursors, and the 4-cycle is not involved in carrying out the same transformation at any other vertex which is not in the neighbourhood,  $\mathcal{N}(v_0)$ , of  $v_0$  given by (16).  $T$  can have at most 3 vertices contained in  $\mathcal{N}(v_0)$  which are in the surface. Therefore, provided that  $\varepsilon(n-1) > (c-1) \times 3$ , there can be at least

$$\binom{\varepsilon(n-1)}{1} \binom{\varepsilon(n-1)-3}{1} \dots \binom{\varepsilon(n-1)-(c-1) \times 3}{1} / c! \tag{35}$$

ways to select the  $c$  vertices at which we can carry out the same transformation to convert  $T$  into a *c*-animal with  $(n+3c)$  vertices. Similarly, we can establish that

$$A_{n+3c}(c, \alpha, \omega) \geq f_3(\alpha, \omega)(\varepsilon n)^c T_n^{(3)}(\varepsilon, >, \alpha, \omega). \tag{36}$$



Combining it with (33) and (34) yields

$$A_{n+3c}(c, \alpha, \omega) \geq f'(\alpha, \omega)(\varepsilon'n)^c T_n(\varepsilon, >, \alpha, \omega) \tag{37}$$

where  $T_n(\varepsilon, >, \alpha, \omega)$  is the partition function for the trees in the set

$$\mathcal{T}_n(\varepsilon, >) = \bigcup \mathcal{T}_n^{(1)}(\varepsilon, >) \bigcup \mathcal{T}_n^{(2)}(\varepsilon, >) \bigcup \mathcal{T}_n^{(3)}(\varepsilon, >). \tag{38}$$

Then by following the same procedure in section 2, we can show that, for a sufficiently small  $\varepsilon$  and sufficiently large  $n$ ,

$$T_n(\varepsilon, >, \alpha, \omega) = T_n(\alpha, \omega)(1 + o(\ )) \tag{39}$$

and therefore

$$A_{n+3c}(c, \alpha, \omega) \geq f'(c, \alpha, \omega)n^c T_n(\alpha, \omega). \tag{40}$$

Combining equations (14) and (40), we obtain theorem 4.

**Theorem 4.** If  $\lim_{n \rightarrow \infty} [\ln T_n(\alpha, \omega) - n \ln \lambda_0(\alpha, \omega)] / \ln n = -\theta_0(\alpha, \omega)$  exists, then

$$\lim_{n \rightarrow \infty} \frac{[\ln A_n(c, \alpha, \omega) - n \ln \lambda_0(\alpha, \omega)]}{\ln n} = -\theta_c(\alpha, \omega) \tag{41}$$

exists and satisfies

$$\theta_c(\alpha, \omega) = \theta_0(\alpha, \omega) - c. \tag{42}$$

#### 4. Discussion

We have studied two lattice models on the square lattice:  $c$ -animals with a nearest-neighbour interaction and  $c$ -animals with a nearest-neighbour interaction and an interaction with an adsorption surface. The result of equation (1) given in sw has been generalized to these two models by establishing the inequalities (11) and (12), which are generalizations of the corresponding inequality

$$a_{n+c}(c) \geq An^c t_n \tag{43}$$

(equation (1.9) in sw). This inequality was previously obtained by showing that cycles can be introduced into a tree at a branch point to give distinct animals with cycles; then for some  $\varepsilon > 0$ , the number of trees with more than  $\varepsilon n$  branch points is equivalent to the number of trees as  $n \rightarrow \infty$  (see lemmas 4–6 in sw). For the two models we have considered, it appears difficult to follow this procedure directly to obtain (11) and (12), since we are unable to show that the partition functions for trees with more than  $\varepsilon n$  branch points are equivalent to the corresponding partition functions for trees. These difficulties have been overcome by showing that cycles can also be introduced into trees at a vertex with nearest-neighbour contacts and at a vertex in an adsorption surface.

The proof given for the square lattice can be generalized to the  $d$ -dimensional hypercubic lattice. For instance, we consider  $c$ -animals with only nearest-neighbour interaction in a  $d$ -dimensional hypercubic lattice. A vertex  $v_0 = (x, y, \dots, z)$  can then have contacts with the vertices  $v_1 = (x + 1, y, \dots, z)$ ,  $v_2 = (x, y + 1, \dots, z)$ ,  $\dots$ ,  $v_d = (x, y, \dots, z + 1)$ . We first convert  $v_0$  into a vertex which is connected to two of its nearest-neighbour vertices and then into a 4-cycle. The resulting animal can have a

finite number of such trees as its precursors. The 4-cycle is totally independent of the same transformation carried out at any other vertex which is not in the neighbourhood,  $\mathcal{N}(v_0)$  defined by

$$\mathcal{N}(v_0): \quad |x - x_0| \leq 1 \quad |y - y_0| \leq 1, \dots, |z - z_0| \leq 1. \quad (44)$$

By following the procedure in section 2, we can establish (9) for c-animals in a  $d$ -dimensional hypercubic lattice.

### Acknowledgments

DZ is grateful to the University of Western Ontario for a Graduate Research Award during Summer 1991. This work was partially supported by NSERC of Canada.

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